Text Supplementary 1 - Imperfect vaccine aggravates the long-standing dilemma of voluntary vaccination

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Dynamics analysis

In the following, we show the dynamics analysis for

$$\dot{x} = x(1-x)(ef(ex)C - V) \tag{1}$$

where the function of infection risk is given by

$$f(x) = \begin{cases} 1 - \frac{1}{R_0(1-x)} & \text{if } 0 \le x < 1 - \frac{1}{R_0} \\ 0 & x \ge 1 - \frac{1}{R_0} \end{cases} \tag{2}$$

Let $G(x) = ef(ex) - r$, where $r = V/C$. The above replicator equation is equivalent to

$$\dot{x} = x(1-x)G(x) \tag{3}$$

under a time rescaling [1].

Theorem For the evolutionary outcome depicted by the replicator equation, we have

(i) When $R_0 \le \frac{1}{1-r}$; all are unvaccinated.

(ii) When $\frac{1}{1-r} < R_0 \le (\frac{1}{1-\sqrt{r}})^2$; if $e \le \frac{r}{1-\frac{1}{R_0}}$, all are unvaccinated, otherwise there is a unique internal stable equilibrium x^*.

(iii) When $R_0 > (\frac{1}{1-\sqrt{r}})^2$; if $e \le \frac{r}{1-\frac{1}{R_0}}$, all are unvaccinated, if $\frac{r}{1-\frac{1}{R_0}} < e \le e_1^*$, there is a unique internal stable equilibrium x^*, if $e_1^* < e \le e_2^*$, full vaccination, if $e_2^* < e$, there is a unique internal stable equilibrium x^*.

Where $x^* = \frac{1-\frac{1}{R_0(1-\frac{r}{e})}}{e}$, $e_{1,2}^* = \frac{1+r}{2} - \frac{1}{2R_0} \pm \frac{\sqrt{R_0^2(1-r)^2 - 2R_0(1+r)+1}}{2R_0}$.

proof

For (i):

since $f(y)$ is decreasing, thus $G(x)$ is decreasing. Therefore if $G(0) < 0$, then $G(x) < 0$ for all the x between 0 and 1. This induces $x^* = 0$ is the unique stable equilibrium for the dynamical system.

But $G(0) = ef(0) - r$. If $R_0 \leq \frac{1}{1-r}$ is valid, then $G(0) = e(1 - \frac{1}{R_0}) - r < 0$ is valid for any e between 0 and 1. This completes the proof for (i).

For (ii):

for $R_0 \leq (\frac{1}{1-\sqrt{r}})^2$, we have that $G(1)$ is smaller than zero for any e. For $R_0 > \frac{1}{1-r}$, the sign of $G(0)$ depends on e: if $e \leq \frac{r}{1-\frac{1}{R_0}}$, it is negative, which results in all unvaccination eventually. Otherwise, it is positive. In this case, there is a unique equilibrium x^* lying between 0 and 1 since $G(x)$ is continues and monotonic [2]. In addition to this, the derivative of $G(x)$ at x^* is negative since $f(x)$ is decreasing. Therefore, there is a unique internal stable equilibrium x^*. By the $f(x)$ we adopt, we have $x^* = \frac{1-\frac{1}{R_0(1-\frac{r}{e})}}{e}$.

This completes the proof of (ii).

For (iii):

if $e \leq \frac{r}{1-\frac{1}{R_0}}$, we have $G(0) < 0$. In analogy to the proof in (i), we have $x^* = 0$ is the unique stable equilibrium.

If $\frac{r}{1-\frac{1}{R_0}} < e \leq e_1^*$, then $G(0) > 0$ while $G(1) < 0$. In analogy to (ii), there is a unique internal stable equilibrium x^*;

If $e_1^* < e \leq e_2^*$, we have $G(1) > 0$, since $G(x)$ is decreasing, we have $G(x)$ is positive for all the x lying between zero and one. Thus, all vaccination becomes the unique stable equilibrium this time.

If $e_2^* < e$, then $G(0) > 0$ while $G(1) < 0$, In analogy to (ii), there is a unique internal stable equilibrium x^*.

This completes the proof.

References

1. Hofbauer J, Sigmund K (1998) Evolutionary Games and Population Dynamics. Cambridge: Cambridge University Press.

2. Rudin W (1986) Principles of mathematical analysis. McGraw-Hill.