The Advantages of a Tapered Whisker

C. M. Williams & E. M. Kramer

Supplemental File 1

Beam theory equations

We follow standard treatments of elastic beam theory [1-3]. Consider an elastic beam deflected by a frictionless pin (Fig. A1). The midline of the beam is parameterized by the arclength s, with $s = 0$ at the origin of the x-y plane. The beam is deflected by a frictionless pin that makes contact with the beam at $s = s_p$. The deflection of the beam is completely characterized by the function $\theta(s)$, to be determined.

Consider a finite segment of the beam, between the origin and s (Fig. A1). The equations describing force and rotational equilibrium on this segment are respectively

$$N_0 = N(s)\cos(\theta(s)) + F(s)\sin(\theta(s)) \tag{A1}$$

$$F_0 = F(s)\cos(\theta(s)) - N(s)\sin(\theta(s)) \tag{A2}$$

$$M(s) = M_0 - xF_0 - yN_0 \tag{A3}$$

where $N(s)$ is the force tangent to the beam at s, $F(s)$ is the shear force, $M(s)$ is the bending moment (calculated about the point s), and the subscript 0 indicates values at the origin. It is helpful to solve Eqns. (A1) and (A2) for N and F

$$N(s) = N_0\cos(\theta(s)) - F_0\sin(\theta(s)) \tag{A4}$$

$$F(s) = F_0\cos(\theta(s)) + N_0\sin(\theta(s)) \tag{A5}$$

The coordinates x and y can be expressed as integrals of $\theta(s)$ via

$$
\begin{aligned}
x(s) &= \int_0^s ds' \cos(\theta(s')) \\
y(s) &= \int_0^s ds' \sin(\theta(s'))
\end{aligned} \tag{A6}
$$

Assuming the pin is frictionless, it can apply a shear force at the contact point, but no tangential force or bending moment. Thus, at the pin (s=s_P), we have the boundary conditions

$$N_P = 0 = N_0 \cos(\theta_P) - F_0 \sin(\theta_P) \tag{A7}$$

$$F_P = F_0 \cos(\theta_P) + N_0 \sin(\theta_P) \tag{A8}$$

$$M_P = 0 = M_0 - x_P F_0 - y_P N_0 \tag{A9}$$

The shear force applied by the pin, F_P is unknown, and is solved for as part of a nonlinear eigenvalue problem.

Lastly, we have the relationship between curvature and moment

$$\frac{d\theta}{ds} = \frac{M(s)}{EI} \tag{A10}$$

where E is the Young's modulus and I is the area moment of inertia. For a beam with circular cross section of radius R, $I = \pi R^4/4$.

Rewriting Eqn. (A7) gives $N_0 = F_0 \tan(\theta_P)$. Then, using (A8) and (A9) to eliminate F_0 and M_0 in favor of F_P, and substituting into (A3) gives

$$M(s) = F_P \left\{ \left[x_P - x(s) \right] \cos(\theta_P) + \left[y_P - y(s) \right] \sin(\theta_P) \right\} \tag{A11}$$

substituting into Eqn. (A10) gives, at last

$$\frac{d\theta}{ds} = \frac{4}{\pi} \frac{F_P}{ER^4} \left\{ \left[x_P - x(s) \right] \cos(\theta_P) + \left[y_P - y(s) \right] \sin(\theta_P) \right\} \tag{A12}$$

Eqns. (A6) and (A12) constitute the set of equations to be solved numerically for $\theta(s)$, subject to the starting boundary condition $\theta(0)=0$. For an untapered beam, R is independent of s. For a tapered beam, $R(s) = R_B(1 - \gamma(s/L))$, where $\gamma = 1 - (R_T/R_B)$, R_B is the radius at the base of the beam, R_T is the radius at the tip of the beam and, and L is the total length of the beam.

From the deflection experiments, we know that the untapered beam maintains $\theta <$ 90° as the pin is dragged past. Thus, for a given x value for the pin (see Fig. 2), the largest y value will occur when the far end of the beam is exactly at the pin (i.e. when $s_P = L$).

Eqns. (A6) and (A12) for a frictionless beam can be simplified by expressing them in terms of the dimensionless variables $u = x/L$, $v = y/L$ and $w = s/L$

$$u(w) = \int_0^w dw' \cos(\theta(w'))$$

$$v(w) = \int_0^w dw' \sin(\theta(w'))$$

$$\frac{d\theta}{dw} = \frac{A}{(1-\alpha w)^4}\left\{\left[u_P - u(w)\right]\cos(\theta_P) + \left[v_P - v(w)\right]\sin(\theta_P)\right\} \tag{A13}$$

where $A = 4F_P L^2/(\pi E R_B^4)$ is the dimensionless shear force at the pin. For a given taper, characterized by α, this presents a nonlinear eigenvalue problem for the set of four parameters (A, u_P, v_P, θ_P).

We solve the set of Eqns. (A13) for the untapered case ($\gamma = 0$), using standard shooting methods [4]. For a given choice of A, there is a unique set (u_P, v_P, θ_P) that satisfies the boundary condition with the pin at the tip of the beam, $w_P = 1$. Having solved this problem for a range of A values, we express the maximum scaled deflection $v_{P,max}$ directly in terms of the scaled contact distance u_P. In Fig. 2 we plot the maximum deflection angle $\theta_{max} = \text{atan}(v_{P,max}/u_P)$ vs. u_P. In Fig. 3 we plot θ_{max} vs. the scaled distance to the pin, $[(u_P)^2+(v_{P,max})^2]^{1/2}$.

Other choices for γ, not solved in this paper, will give different solutions for the eigenvalue problem. Thus, the maximum deflection and protraction angles, plotted as a function of scaled contact distance, depend on the taper through γ, and on friction with the pin, but are otherwise independent of E, R_B, and L.

References

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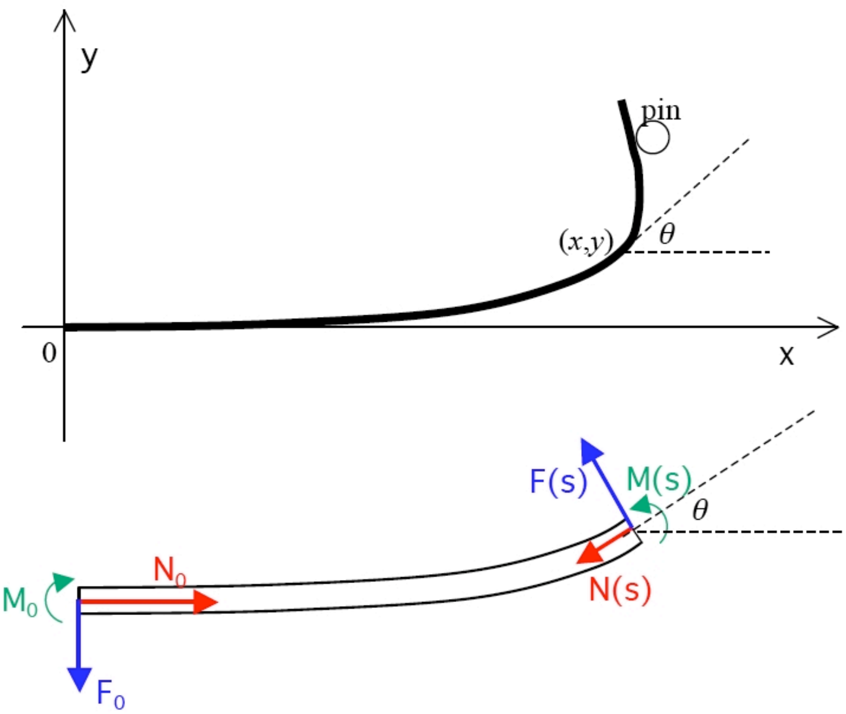

Figure A1. Elastic beam. Top: coordinates used in the calculation. θ is the angle between the beam tangent vector and the x-axis. Bottom: A segment of the beam, between the origin and s, showing the forces and moments applied to the ends. Red: normal forces. Blue: shear forces. Green: bending moments. Note that the bending moment at s is defined to be positive if it produces an upward curvature.