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A Random Matrix Approach to Credit Risk

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Abstract

We estimate generic statistical properties of a structural credit risk model by considering an ensemble of correlation matrices. This ensemble is set up by Random Matrix Theory. We demonstrate analytically that the presence of correlations severely limits the effect of diversification in a credit portfolio if the correlations are not identically zero. The existence of correlations alters the tails of the loss distribution considerably, even if their average is zero. Under the assumption of randomly fluctuating correlations, a lower bound for the estimation of the loss distribution is provided.

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Introduction

The financial crisis of 2008–2009 clearly revealed that an improper estimation of credit risk can lead to dramatic effects on the world's economy. The vast underestimation of risks embedded in credits for the subprime housing markets induced a chain reaction that propagated into the worldwide economy. A better estimation of credit risk (see, eg, [1,2,3,4,5]) is therefore of vital interest. We can distinguish two fundamentally different approaches to credit risk modeling (see, eg, [6]): the structural and the reduced–form approach.

Structural models have a long history, going back to the work of Black and Scholes [7] and Merton [8]. The *Merton model* assumes a zero–coupon debt structure with a fixed time to maturity. The value of the company's assets is modeled by a stochastic process. The risk of default and the associated recovery rate, the residual payment in case of a loss, are directly determined by the company's asset value at maturity.

Reduced–form models attempt to capture the dependence of default and recovery rates on macroeconomic risk factors. Both quantities are modeled as independent stochastic variables. Some well known reduced–form model approaches can be found in [9,10,11,12,13].

First passage models were first introduced by Black and Cox [14] and they fall somewhat in between the two modeling approaches. Similar to Merton's model, the market value of a company is modeled by a stochastic process. However, in the first passage models a default occurs whenever this market value hits a certain threshold for the first time. The recovery rates are typically modeled independently, for example, by a reduced–form model, see [15,16], or are even assumed to be constant, see, eg, [6]. Recent approaches aim at improving first passage models by including the chance of full recovery, even if a company's market value is below the threshold, see [17], and estimating correlations between default probabilities of industry sectors, see [18].

Reduced–form and first–passage models are implemented in commercial software solutions, for example, *CreditMetrics* initially developed by JP Morgan [19], *CreditPortfolioView* by McKinsey & Company [20] or *CreditRisk+* by Credit Suisse [21]. As there can be a strong connection between default risks and recovery rates, the chances of large losses are often underestimated in the reduced–form and first passage models, see [22,23]. The Merton model does not require this separation and is, for example, adopted by Moody's KMV.

Structural models provide a "microscopical'' tool to study credit risk as the defaults and recoveries are traced back to stochastic processes modeling the state of individual obligors. For a portfolio of credits, such as collateralized debt obligations (CDOs), correlations represent a key factor that influences its risk. The benefit of diversification, ie, the reduction of risk by increasing the portfolio size, is severely limited by the presence of even weak correlations. This has been demonstrated for the case of constant positive correlations, both in the first passage model with constant recovery [24,25] and in the Merton model [26,22]. The key problem in estimating the credit risk of a realistic portfolio is of course the huge number of parameters involved. This is precisely where approaches from statistical physics can be most helpful: the state of a system with many degrees of freedom is, under certain conditions, described by few macroscopic observables. In the thermodynamic equilibrium, these are energy, temperature, pressure, etc. Ergodicity holds, ie, time and ensemble average yield the same results. A somewhat similar situation exists for spectral statistics in quantum chaotic systems, see [27]. A moving average over one long spectrum equals an ensemble average over random matrices, if the number of levels is very large. Originally, random matrix theory was developed in the 1950s to describe the spectra of heavy nuclei, see [28]. Here we transfer this idea to large credit portfolios involving correlated assets. In the case of a great many contracts, we expect a self–averaging property which then should allow to average over an ensemble of random correlation matrices. We manage to carry out this approach largely analytically. We obtain estimates for the distribution of asset values and the portfolio loss distribution in which the complicated effects of all correlations are indeed reduced to a single parameter measuring the correlation strength.

A Structural Credit Risk Model

Our model is based on Merton's original model, assuming a zero-coupon bond for the debt structure of the obligor. Our aim is to analytically describe the impact of correlations on the losses of a credit portfolio. Even though the Merton model makes many simplifying assumptions, it can provide more than just qualitative insights into credit risk. Indeed we demonstrated recently that empirical credit data are in accordance with analytical results derived from the Merton model [29].

The cash flow of the zero-coupon bond is limited to two dates: the date of issue $t=0$ and maturity $t=T$. At the issue date the creditor lends a specified amount of money to the obligor. At maturity, the obligor has to repay the face value of the bond. The face value is the amount borrowed plus interest and risk premium. A default occurs if the asset value V_k of company k is below the face value F_k at maturity time T. The size of the loss then depends on how far V_k is below the face value F_k. We assume that the asset values in a portfolio of K companies follow a geometric Brownian motion. An overview of the model's input parameters is given in Table 1.

Average distribution of asset values

For the sake of simplicity, let us first consider the case of a Brownian motion for the asset values. Later on this can be easily mapped to the geometric Brownian motion by a simple substitution. For a Brownian motion, the probability density function (pdf) of the vector V of K asset values at maturity T is described by

$$p^{(\mathrm{mv})}(V,\Sigma)=\frac{1}{\sqrt{2\pi T}^{K}}\frac{1}{\sqrt{\det(\Sigma)}}\exp\left(-\frac{1}{2T}(V-\mu T)^{\dagger}\Sigma^{-1}(V-\mu T)\right)\tag{1}$$

Here, Σ is the covariance matrix and μ is the drift vector. For later convenience we can express this as a Fourier transform,

$$p^{(\mathrm{mv})}(V,\Sigma)=\frac{1}{(2\pi)^{K}}\int\exp\left(-i\omega^{\dagger}(V-\mu T)\right)\exp\left(-\frac{T}{2}\omega^{\dagger}\Sigma\omega\right)\mathrm{d}[\omega]\tag{2}$$

Equation (1) gives the pdf of asset values in the case of a correlated Brownian motion. However, we are not interested in the impact of a specific correlation matrix. Instead we want to estimate the general impact of correlations. To this end, we want to average over all possible correlation matrices and disclose the general statistical behavior.

We use a random matrix approach to calculate the average distribution of asset values, $\langle p^{(\mathrm{mv})}(V)\rangle$, for random correlations where the average correlation level is zero. To achieve this we replace the covariance matrix Σ by

$$\Sigma_W=SWW^{\dagger}S\tag{3}$$

where $S=\mathrm{diag}(\sigma_1,\ldots,\sigma_K)$ contains the standard deviations and $W\in\mathbb{R}^{K\times N}$ is a random matrix. The entries of W are independent and Gaussian distributed,

$$p^{(\mathrm{corr})}(W)=\sqrt{\frac{N}{2\pi}}^{KN}\exp\left(-\frac{N}{2}\mathrm{tr}\,W^{\dagger}W\right)\tag{4}$$

with variance $1/N$. The resulting correlation matrix $WW^{\dagger}$ is Wishart-distributed [30] with average correlation zero. With the parameter N we can control how strongly the entries of $WW^{\dagger}$ fluctuate. For $N\to\infty$, we obtain the unit matrix for $WW^{\dagger}$, ie, the uncorrelated case. For $N\geq K$, we obtain an invertible covariance matrix with random entries. The case $N<K$ is disregarded as the resulting matrix is not invertible which is usually required for applications in risk management. When inserting this ansatz into Eq. (2), we obtain

$$\langle p^{(\mathrm{mv})}(V)\rangle=\int p^{(\mathrm{corr})}(W)p^{(\mathrm{mv})}\left(V,SWW^{\dagger}S\right)\mathrm{d}[W]\tag{5}$$

$$=\frac{\sqrt{N}^{NK}}{(2\pi)^{K}}\int\exp\left(-i\omega^{\dagger}V\right)\frac{1}{\sqrt{\det(NI+TS\omega\omega^{\dagger}S)}^{N}}\mathrm{d}[\omega]\tag{6}$$

Table 1. Input of the structural credit risk model.

Variable	Description	Unit
K	Number of contracts	–
T	Time to maturity	[year]
σ_k	Volatility of asset k	$[\mathrm{year}]^{-1/2}$
μ_k	Drift of asset k	$[\mathrm{year}]^{-1}$
N	Parameter to control correlations,	–
	$N\to\infty$: uncorrelated limit	
$V_{K,0}$	Initial value of asset k	[currency]
F_k	Face value of contract k	[currency]

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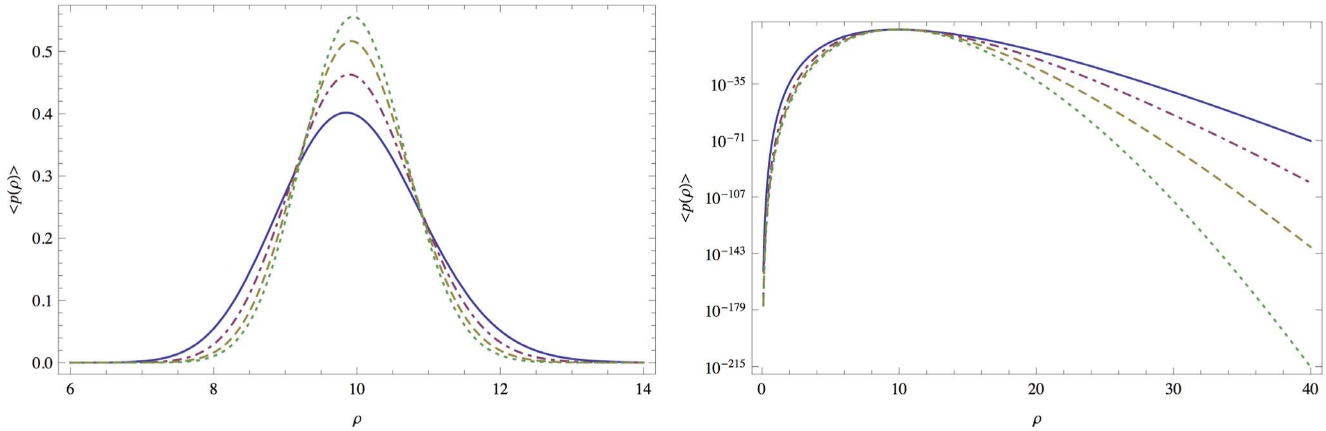

(a) linear scale **(b)** logarithmic scale

Figure 1. Illustration of the average asset value distribution $\langle p^{(\mathrm{mv})}(\rho)\rangle$ for $T = 1$, K = 50 and different values for N. Solid, dashed{dotted, dashed and dotted lines correspond to $N = K$, $2K$, $5K$ and $30K$, respectively.

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where I denotes the unit matrix. A detailed derivation is given in appendix S1. Here we choose $\mu = 0$. We will reintroduce the drift later on, when we make the substitution for the geometric Brownian motion. The determinant can be written as

$$\det(NI + TS\omega\omega^\dagger S) = N^K \left(1 + \frac{T}{N}\omega^\dagger SS\omega\right) \tag{7}$$

because the matrix $S\omega\omega^\dagger S$ has rank one. Hence, we arrive at

$$\langle p^{(\mathrm{mv})}(V)\rangle = \frac{1}{(2\pi)^K}\int \exp\left(-i\omega^\dagger V\right) \frac{1}{(1+(T/N)\omega^\dagger SS\omega)^{N/2}}\mathrm{d}[\omega] \tag{8}$$

This integral can be calculated by using the Gamma function (see [31]) in the form

$$\frac{\Gamma(x)}{a^x} = \int\limits_0^\infty z^{x-1}\exp(-az)\mathrm{d}z, \quad x>0,\, a>0 \tag{9}$$

We identify a^{-x} with $(1+(T/N)\omega^\dagger SS\omega)^{-N/2}$ and obtain

$$\langle p^{(\mathrm{mv})}(V)\rangle = \frac{1}{(2\pi)^K}\frac{1}{\Gamma(N/2)}\left(\prod_{k=1}^K \frac{1}{\sigma_k}\right) \times \int\limits_0^\infty z^{\frac{N}{2}-1}\exp(-z)\sqrt{\frac{\pi N}{zT}}^K \exp\left(-\frac{N}{4Tz}\sum_{k=1}^K \frac{V_k^2}{\sigma_k^2}\right)\mathrm{d}z \tag{10}$$

as worked out in appendix S2. This integral is a representation of the Bessel function of the second kind $\mathcal{K}$ of the order $(K-N)/2$, see [32]. Thus, we obtain

$$\langle p^{(\mathrm{mv})}(V)\rangle = \sqrt{\frac{N}{2\pi T}}^K \frac{2^{1-\frac{N}{2}}}{\Gamma(N/2)}\left(\prod_{k=1}^K \frac{1}{\sigma_k}\right) \sqrt{\frac{N}{T}\sum_{k=1}^K \frac{V_k^2}{\sigma_k^2}}^{\frac{N-K}{2}} \mathcal{K}_{\frac{N-K}{2}}\left(\sqrt{\frac{N}{T}\sum_{k=1}^K \frac{V_k^2}{\sigma_k^2}}\right) \tag{11}$$

for the average distribution of $p^{(\mathrm{mv})}(V)$ if assuming a randomly distributed correlation matrix and an average correlation level of zero. We stated earlier that we include N in the distribution of the random matrices W in order to render the variance of the average asset value distribution N-independent. The variances only depend on T and σ_k, as discussed in appendix S3. The parameter N is only used to control the correlations. In hyperspherical coordinates, Equation 11 depends only on the hyperradius

$$\rho \equiv \sqrt{\sum_{k=1}^K \frac{V_k^2}{\sigma_k^2}} \tag{12}$$

This leads to the expression

$$\langle p^{(\mathrm{mv})}(\rho)\rangle = \sqrt{\frac{N}{2\pi T}}^K \frac{2^{1-\frac{N}{2}}}{\Gamma(N/2)}\rho^{\frac{N+K-1}{2}} \sqrt{\frac{N}{T}}^{\frac{N-K}{2}} \mathcal{K}_{\frac{N-K}{2}}\left(\rho\sqrt{\frac{N}{T}}\right) \tag{13}$$

for the hyperradial density function, cf. appendix S3. We illustrate this density function in Figure 1 for $K = 50$ and different values of N. We note that the tail-behavior for large ρ is exponential.

We obtain the average asset value distribution in case of a geometric Brownian motion by a simple substitution $V_k \to \hat{V}_k$,

$$\langle p^{(\mathrm{mv})}(V)\rangle = \sqrt{\frac{N}{2\pi T}}^{K} \frac{2^{1-\frac{N}{2}}}{\Gamma(N/2)} \left(\prod_{k=1}^{K} \frac{1}{\sigma_k V_k}\right) \sqrt{\frac{N}{T}\sum_{k=1}^{K}\frac{\hat{V}_k{}^2}{\sigma_k^2}}^{\frac{N-K}{2}} \mathcal{K}_{\frac{N-K}{2}}\left(\sqrt{\frac{N}{T}\sum_{k=1}^{K}\frac{\hat{V}_k{}^2}{\sigma_k^2}}\right) \tag{14}$$

with

$$\hat{V}_k = \ln\left(\frac{V_k}{V_{k,0}}\right) - \left(\mu_k - \frac{\sigma_k^2}{2}\right)T \tag{15}$$

Here, the parameter σ_k refers to the standard deviation of the underlying Brownian motion, ie, the volatility of asset returns. The resulting asset values thus have the variance

$$\hat{\sigma}_k^2 = \exp(2\mu_k T)\left(\exp\left(\sigma_k^2 T\right) - 1\right) V_{k,0}^2 \tag{16}$$

where $V_{k,0}$ are the starting asset values at $t=0$. Figure 2 shows the distribution of asset values based on a geometric Brownian motion, as given in Eq. (14). The findings are similar to the case of the Brownian motion. While we obtain a narrow but heavy-tailed distribution for $N=K$, the distribution slowly approaches an uncorrelated bivariate log-normal distribution with increasing values of N.

Loss distribution

We now turn to the calculation of the loss distribution. A default occurs if the asset value V_k at maturity T is lower than the face value F_k. The size of the loss is given by the difference of F_k and V_k. Even if a loss occurs, the creditor might not lose all money that he lent, because the obligor is still able to pay back the amount V_k. In order to compare losses in a portfolio of credits, we have to normalize them by the corresponding face value. We define the normalized loss L_k of the k-th asset as

$$L_k = \begin{cases} \frac{F_k - V_k}{F_k}, & V_k < F_k \quad \text{(default)} \\ 0, & \text{else} \quad \text{(no default)} \end{cases} \tag{17}$$

We observe that the asset values have to be positive in Eq. (17). Therefore we assume in all further considerations that the underlying asset value process is given by a geometric Brownian motion.

When calculating the overall loss of a portfolio, we have to weight each loss by its face value in relation to the sum of all portfolio face values,

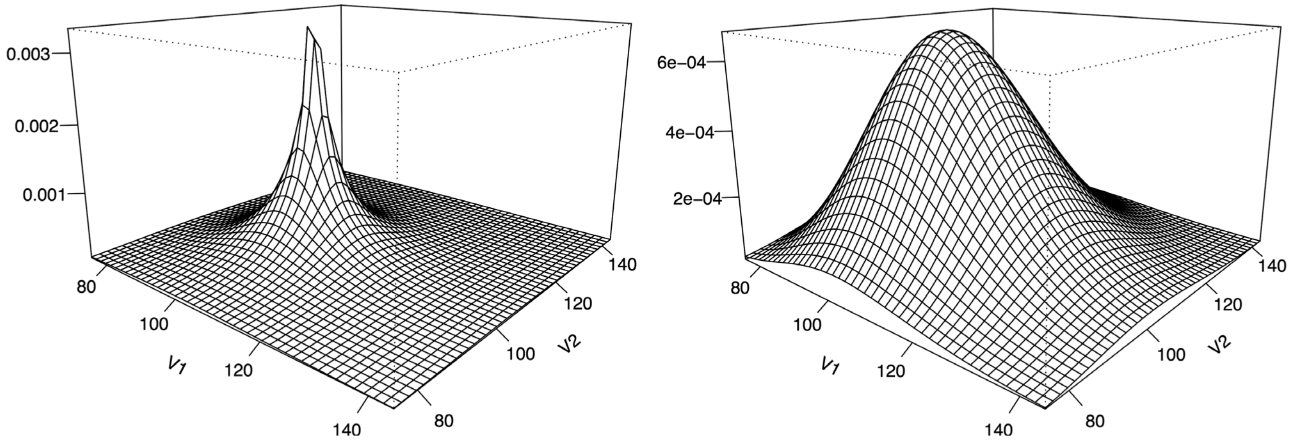
$$L = \sum_{k=1}^{K} f_k L_k, \quad f_k = \frac{F_k}{\sum_{i=1}^{K} F_i} \tag{18}$$

We integrate over the pdf of asset values and filter for those that lead to a given total loss L. By the above stated definitions, we can define a filter for the total loss at maturity time T. In the next step we express the filter using a Fourier transformation. Eventually, we separate those terms that correspond to a default and those that describe the asset values above the face value F_k.

$$p^{(\mathrm{loss})}(L) = \int_0^{\infty} \mathrm{d}[V] p^{(\mathrm{mv})}(V) \delta\left(L - \sum_{k=1}^{K} f_k L_k\right) \tag{19}$$

$$= \int_0^{\infty} \mathrm{d}[V] p^{(\mathrm{mv})}(V) \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{d}v \exp\left(-ivL + iv\sum_{k=1}^{K} f_k L_k\right) \tag{20}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{d}v \exp(-ivL) \int_0^{\infty} \mathrm{d}[V] \exp\left(iv\sum_{k=1}^{K} f_k L_k\right) p^{(\mathrm{mv})}(V) \tag{21}$$



(a) $N=2$ **(b)** $N=100$

Figure 2. Illustration of the average asset value distribution $\langle p^{(\mathrm{mv})}(V)\rangle$ with a geometric Brownian motion for $T=1$, $K=2$, $V_{k,0}=100$, $\mu=0.05$ and different values for N. Both distributions have the identical standard deviation $\hat{\sigma} \approx 16$ ($\sigma = 0.15$). For $N=2$, we obtain a heavy-tailed distribution while the uncorrelated limit is reached for $N=100$.
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$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{d}\nu \exp(-i\nu L)$$

$$\times \prod_{k=1}^{K} \left[\int_{0}^{F_k} \mathrm{d}V_k \exp\left(i\nu f_k \left(1 - \frac{V_k}{F_k} \right) \right) + \int_{F_k}^{\infty} \mathrm{d}V_k \right] p^{(\mathrm{mv})}(V) \qquad (22)$$

Here, the expression in the square brackets acts as an operator, because $p^{(\mathrm{mv})}(V)$ does not necessarily factorize. We will use this ansatz to calculate the average loss distribution in the next section. However, Eq. (22) can be used to calculate the loss distribution if the actual asset value distribution is known, ie, the statistical dependence and the underlying process are estimated. To prepare for this, it is handy to write Eq. (22) as a combinatorial sum,

$$p^{(\mathrm{loss})}(L) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{d}\nu \exp(-i\nu L) \qquad (23)$$

$$\times \sum_{k=1}^{K} \sum_{j=1}^{\binom{K}{k}} \left[\begin{array}{l} \prod\limits_{l \in \mathrm{Perm}(j,k,K)} \int_0^{F_l} \mathrm{d}V_l \exp\left(i\nu f_l \frac{F_l - V_l}{F_l} \right) \\ \prod\limits_{\substack{q \in \{1 \ldots K\} \\ \setminus \mathrm{Perm}(j,k,K)}} \int_{F_q}^{\infty} \mathrm{d}V_q \end{array} \right] p^{(\mathrm{mv})}(V)$$

where $\mathrm{Perm}(j,k,K)$ is the j-th permutation of k elements of the set $\{1 \ldots K\}$. For example, if $K=3$ and $k=2$, we obtain, $\mathrm{Perm}(1,2,3) = \{1,2\}$, $\mathrm{Perm}(2,2,3) = \{2,3\}$ and $\mathrm{Perm}(3,2,3) = \{1,3\}$. However, Eq. (24) might need to be estimated numerically, depending on the complexity of the asset value distribution $p^{(\mathrm{mv})}(V)$. In the section *Homogeneous portfolios*, we will simplify this combinatorial sum for a homogeneous portfolio and the average asset value distribution $\langle p^{(\mathrm{mv})}(V) \rangle$.

Average loss distribution

Now we have developed all necessary tools to model the average distribution of losses, under the assumption of random correlations and an average correlation level of zero. We start by inserting the average asset value distribution in a component-wise notation (cf. appendix S2) into the loss distribution (22),

$$\langle p^{(\mathrm{loss})}(L) \rangle = \frac{1}{2\pi \Gamma(N/2)} \int_0^{\infty} \mathrm{d}z \, z^{\frac{N}{2}-1} \exp(-z) \int_{-\infty}^{+\infty} \mathrm{d}\nu \exp(-i\nu L) r(\nu, z) \qquad (24)$$

with

$$r(\nu,z) = \prod_{k=1}^{K} \left[\int_0^{F_k} \mathrm{d}V_k \exp\left(i\nu f_k \frac{F_k - V_k}{F_k} \right) + \int_{F_k}^{\infty} \mathrm{d}V_k \right]$$

$$\times \frac{\sqrt{N}}{2\sigma_k V_k \sqrt{\pi z T}} \exp\left(-\frac{N(\ln(V_k/V_{k,0}) - (\mu_k - \sigma_k^2/2)T)^2}{4zT\sigma_k^2} \right) \qquad (25)$$

We carry out a second order approximation of this expression in appendix S4 and arrive at

$$\langle p^{(\mathrm{loss})}(L) \rangle = \frac{1}{\sqrt{2\pi}\Gamma(N/2)} \int_0^{\infty} \mathrm{d}z \, z^{\frac{N}{2}-1} \exp(-z) \frac{1}{\sqrt{\hat{m}_2(z)}} \exp\left(-\frac{(L - \hat{m}_1(z))^2}{2\hat{m}_2(z)} \right) \qquad (26)$$

with

$$\hat{m}_1(z) = \sum_{k=1}^{K} f_k m_{1,k}(z) \qquad (27)$$

$$\hat{m}_2(z) = \sum_{k=1}^{K} f_k^2 (m_{2,k}(z) - m_{1,k}(z)^2) \qquad (28)$$

and

$$m_{j,k}(z) = \frac{\sqrt{N}}{2\sigma_k \sqrt{\pi z T}} \int_0^{F_k} \frac{1}{V_k} \left(\frac{F_k - V_k}{F_k} \right)^j$$

$$\times \exp\left(-\frac{N(\ln(V_k/V_{k,0}) - (\mu_k - \sigma_k^2/2)T)^2}{4zT\sigma_k^2} \right) \mathrm{d}V_k \qquad (29)$$

However, the convergence radius of the power series expansion involved in this approximation is one. Although we consider large portfolios K, ie, f_k is small, ν runs from $-\infty$ to $+\infty$. This second-order approximation might describe the default terms adequately. However, the non-default terms, corresponding to a delta peak at $L=0$ require ν to run from $-\infty$ to $+\infty$. Thus, the non-default terms cannot be approximated using this second-order approximation. To circumvent this problem we develop an improved approximation in the next section.

Due to the complexity of $\hat{m}_1(z)$ and $\hat{m}_2(z)$, the z integral needs to be evaluated numerically. We present this for the example of a homogeneous portfolio.

Homogeneous portfolios

In case of a homogeneous portfolio, in which all credits have the same face value $F_k = F$ and the same variance $\sigma_k^2 = \sigma^2$ and initial value $V_{k,0} = V_0$, the weights can be simplified to

$$f_k = \frac{1}{K} \tag{30}$$

As $m_{1,k}(z)$ and $m_{1,k}(z)$ become identical for every k, we denote them by $m_1(z)$ and $m_1(z)$ leading to

$$\widehat{m}_1(z) = m_1(z) \tag{31}$$

$$\widehat{m}_2(z) = \frac{1}{K}\left(m_2(z) - m_1(z)^2\right) \tag{32}$$

$$m_j(z) = \frac{\sqrt{N}}{2\sigma\sqrt{\pi z T}} \int_0^F \frac{1}{V}\left(\frac{F-V}{F}\right)^j$$

$$\times \exp\left(-\frac{N(\ln(V/V_0) - (\mu - \sigma^2/2)T)^2}{4zT\sigma^2}\right) \mathrm{d}V \tag{33}$$

Here V is a scalar and we only have to calculate a single integral over V. After inserting this into Eq. (26), we can calculate the loss distribution for a homogeneous portfolio in the second order approximation.

Improved approximation for a homogeneous portfolio

The second order approach can be improved by approximating the individual terms of the loss distribution instead of approximating the expression as a whole, similar as discussed in [26]. In case of a homogeneous portfolio the combinatorial sum in Eq. (24) reduces to

$$\langle p^{(\mathrm{loss})}(L)\rangle = \frac{1}{2\pi\Gamma(N/2)} \int_0^\infty \mathrm{d}z\, z^{\frac{N}{2}-1} \exp(-z) \int_{-\infty}^{+\infty} \mathrm{d}v \exp(-ivL)$$

$$\times \sum_{j=0}^{K} \binom{K}{j} \left(r^{(\mathrm{D})}(v,z)\right)^j \left(r^{(\mathrm{ND})}(z)\right)^{K-j} \tag{34}$$

with the non-default term $\left(r^{(\mathrm{ND})}\right)^{K-j}$ where

$$r^{(\mathrm{ND})} = \int_F^\infty \mathrm{d}V \frac{\sqrt{N}}{2\sigma V\sqrt{\pi z T}}$$

$$\times \exp\left(-\frac{N(\ln(V/V_0) - (\mu - \sigma^2/2)T)^2}{4zT\sigma^2}\right) \tag{35}$$

$$= \frac{1}{2} + \frac{1}{2}\mathrm{Erf}\left(\frac{\sqrt{N}(\ln(F/V_0) - (\mu - \sigma^2/2)T)}{2\sigma\sqrt{zT}}\right) \tag{36}$$

and the default term $\left(r^{(\mathrm{D})}(v,z)\right)^j$ where

$$r^{(\mathrm{D})}(v,z) = \int_0^F \mathrm{d}V \exp\left(\frac{iv}{K}\frac{F-V}{F}\right)$$

$$\times \frac{\sqrt{N}}{2\sigma V\sqrt{\pi z T}} \exp\left(-\frac{N(\ln(V/V_0) - (\mu - \sigma^2/2)T)^2}{4zT\sigma^2}\right) \tag{37}$$

In the homogeneous case, the integration variable V is a scalar. The approximation follows the same principles as in the previous section, resulting in

$$\int_{-\infty}^{+\infty} \mathrm{d}v \exp(-ivL)\left(r^{(\mathrm{D})}(v,z)\right)^j$$

$$= \int_{-\infty}^{+\infty} \mathrm{d}v \exp\left(iv\left(\frac{j}{K}m_1(z) - L\right) - \frac{v^2 j}{2K^2}\left(m_2(z) - m_1(z)^2\right)\right) \tag{38}$$

$$= \sqrt{\frac{2\pi K^2}{j\left(m_2(z) - m_1(z)^2\right)}} \exp\left(-\frac{(LK - jm_1(z))^2}{2j\left(m_2(z) - m_1(z)^2\right)}\right) \tag{39}$$

In this approximation, the non-default terms given by Eq. (36) can be calculated exactly. They correspond to a delta peak at $L = 0$. Another advantage over the approach presented in Eq. (26) is that the approximation is performed for each number of defaults j separately and weighted by j/K accordingly. Here, the omitted third term is of the order j/K^3 and thereby much smaller than the third term of the simple second order approximation (33), which would be of the order $1/K^2$. Thus, when approximating each term in the combinatorial sum separately, we obtain an improved result. Insertion into (34) leads to

$$\langle p^{(\mathrm{loss})}(L)\rangle \approx \frac{1}{2\pi\Gamma(N/2)} \sum_{j=0}^{K} \binom{K}{j} \int_0^\infty \mathrm{d}z\, z^{\frac{N}{2}-1} \exp(-z)$$

$$\times \sqrt{\frac{2\pi K^2}{j\left(m_2(z) - m_1(z)^2\right)}} \exp\left(-\frac{(LK - jm_1(z))^2}{2j\left(m_2(z) - m_1(z)^2\right)}\right)$$

$$\times \left(\frac{1}{2} + \frac{1}{2}\mathrm{Erf}\left[\frac{\ln(F/V_0) - (\mu - \sigma^2/2)T}{2\sigma\sqrt{zT}}\right]\right)^{K-j} \tag{40}$$

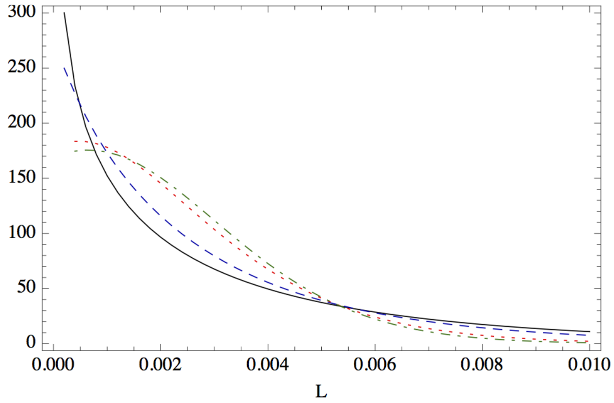

Figure 3. The loss distribution for K = 10, σ = 0.15, μ = 0.05, T = 1, V_0 = 100, F = 75 and different amounts of randomness in the correlation matrix, N = K (solid black), N = 2K (dashed blue), N = 10K (dotted red), N = 30K (dot-dashed green).
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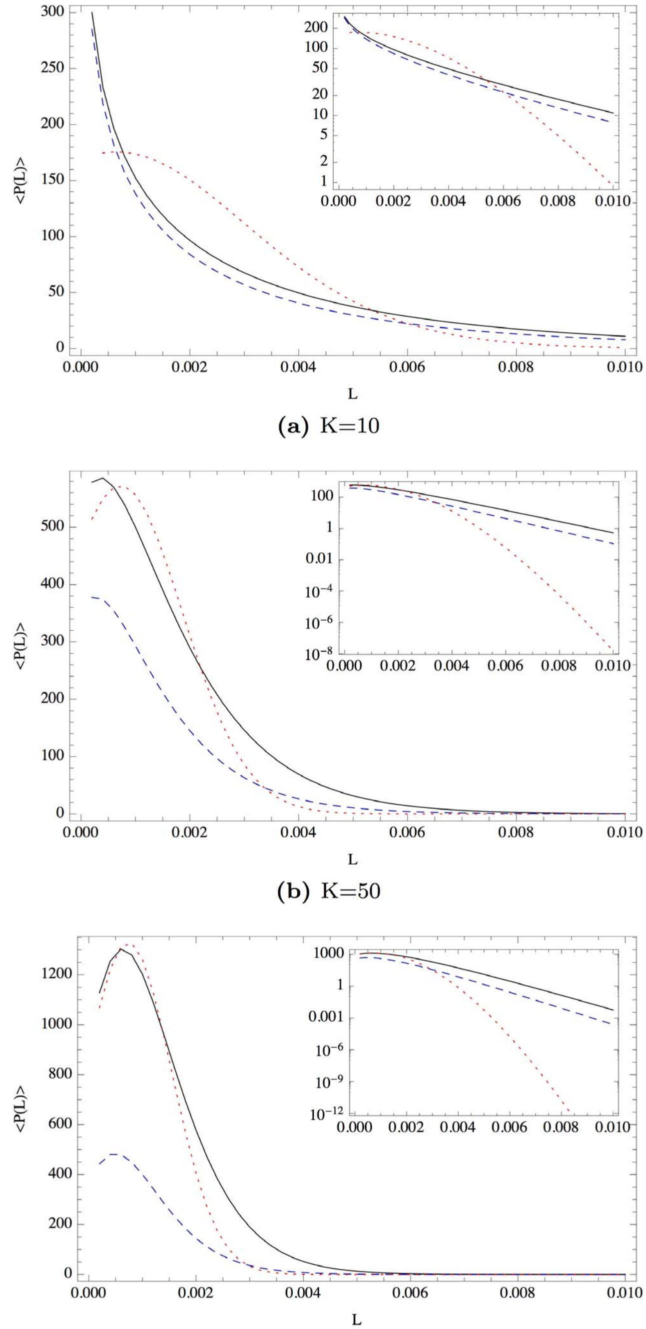
which is the final result.

Results

We now apply the analytically developed model to a specific example. To analyze the impact of correlations, we calculate the loss distribution for different homogeneous portfolios with sizes $K=10$, $K=50$ and $K=100$ with the parameters $V_0=100$, $\mu=0.05$, $\sigma=0.15$, $F=75$ and $T=1$. As stated in the previous section, we can control the amount of correlation in our model with the parameter N. Since we only consider correlation matrices with full rank, we obtain the strongest correlations if we choose $N=K$. For $N\to\infty$, the correlation matrix becomes the unit matrix. Thus, this represents the transition to a system without correlations. As we have to evaluate the loss distributions numerically, the limit $N\to\infty$ has to be properly interpreted. We need to identify a value for which this convergence is valid in good approximation. Figure 3 illustrates the loss distribution for $K=10$ and different values of N. Our study indicates that a value of $N=30K$ is a good choice for approximating the uncorrelated case and is still numerically feasible. The results are presented in Fig. 4. For all portfolio sizes, $K=10$, $K=50$ and $K=100$, we obtain heavier tails of the loss distribution of the correlated portfolio compared to the uncorrelated case. Even the simple approximation, represented by the dashed blue curve, exhibits these heavy tails. With the inserted logarithmic plots, we can identify a nearly power-law decay of the loss distribution for the correlated case.

The distributions become narrower for larger values of K. However, the tails of the correlated case remain heavier than those of the uncorrelated case. While both approximations yield similar results for $K=10$, their difference becomes larger with K. As both approximations have to be performed numerically, the improved approximation is always favored. However, the tail behavior remains the same, even for the simple approximation, as indicated by the logarithmic scaled inserts in Fig. 4. This is a strong indication that the tails of the loss distribution are vastly underestimated if correlations are not taken into account.

Due to the approximation, the normalization of the loss distribution is not exact. Especially the normalization of the simple approximation is problematic for large values of K. The normalization might also be used as an indication for the quality of the approximation. The improved approximation exhibits a delta peak at $L=0$, as the non-default terms can be calculated exactly.



(a) K=10

(b) K=50

(c) K=100

Figure 4. The loss distribution of a homogeneous portfolio with σ = 0.15, μ = 0.05, T = 1, V_0 = 100, F = 75 and different values of K. The blue dashed line represents the simple approximation; the solid black line represents the improved approximation. Both have been calculated for the strongest random correlations, N = K. The uncorrelated case is given by the red dotted line, calculated with the improved approximation with N = 30K.
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However, the interval $[0;0.0002[$ was not evaluated due to numerical feasibility.

In our example, we do not vary the maturity time T, ie, we choose $T=1$. One can increase T to estimate the evolution of the loss distribution. However, this evolution depends strongly on the drifts μ_k and standard deviations σ_k. Depending on their value, the exposure to default risk can either increase or decrease.

Discussion

To assess the risk of a credit portfolio, it is crucial to take correlations between obligors into account. We consider the Merton model, in which defaults and recoveries are determined by the underlying asset processes. The correlation matrix of the asset returns has to be estimated from historical time series. This is not always easy, because the correlations change in time, ie, they are non-stationary. Since only time series up to a certain length can be used, the correlation coefficients contain a specific type of randomness, see [33,34]. Several methods have been put forward to estimate and to reduce this "noise". Thus, we assume that such a noise reduction has been done. The corresponding "true" correlation coefficients and matrices are the proper input for the structural credit risk model of the Merton type that we consider. We discussed this issue of noise reduction to emphasize that the random matrix approach in that context focuses on the spectral statistics of correlation or covariance matrices, see [35,36,37,38,39]. It is based on a very different motivation as compared to the present application.

Searching for generic properties, we devised the present random matrix approach. Instead of calculating the portfolio loss distribution for a specific correlation matrix, we average over an ensemble of random correlation matrices. Our approach transfers concepts of statistical physics. In quantum chaos, the average over an individual, long spectrum equals the average over an ensemble of random matrices, if the level number is very high. We expect that a similar self-averaging property also holds here. This line of reasoning is supported by the following consideration: The correlation coefficients are varying functions in time, because the business relations of the companies change. This implies that a correlation matrix over a somewhat longer period in time is a varying quantity, ie, it corresponds to some kind of ensemble.

In our model the average correlation level is zero and we assume that there is no branch structure in the correlations. The fluctuation strength of individual correlations is controlled by a single parameter. This ansatz allowed us to estimate generic statistical properties of the Merton model. Some features are not taken into account which are present in empirical data, such as jumps or an overall positive correlation level. Those features are difficult to treat completely analytically. However, even in our simple setup we obtain a heavy–tailed loss distribution. In this sense our model can be used to estimate a lower bound of the risk embedded in a credit portfolio.

Our results clearly demonstrate that the risk in a credit portfolio is heavily underestimated if correlations are not taken into account. Even for random correlations with an average correlation level of zero, we observe very slowly decaying portfolio loss distributions. In contrast, the probability of large losses in uncorrelated portfolios is significantly reduced within the Merton model.

The results are especially relevant for CDOs, bundles of credits that are traded on equity markets. CDOs are constructed in order to lower the overall risk. The components of a CDO can be exposed to large risks. It is often believed that the CDO has a significantly lower risk. We showed that this diversification only works well if the correlations in the credit portfolio are identical to zero.

Supporting Information

Appendix S1.
(PDF)

Appendix S2.
(PDF)

Appendix S3.
(PDF)

Appendix S4.
(PDF)

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Author Contributions

Conceived and designed the experiments: MCM RS TG. Performed the experiments: MCM RS. Analyzed the data: MCM RS. Contributed reagents/materials/analysis tools: MCM RS TG. Wrote the paper: MCM RS.

References

1. Bluhm C, Overbeck L, Wagner C (2002) An Introduction to Credit Risk Modeling. Taylor and FrancisCRC Press
2. Bielecki TR, Rutkowski M (2005) Credit Risk: Modeling, Valuation and Hedging. Springer.
3. Duffie D, Singleton KJ (2003) Credit Risk: Pricing, Measurement, and Management. Princeton University Press.
4. Lando D (2004) Credit Risk Modeling: Theory and Applications. Princeton University Press
5. McNeil AJ, Frey R, Embrechts P (2005) Quantitative Risk Management: Concepts, Techniques, and Tools. Princeton University Press
6. Giesecke K (2004) Credit Risk: Models and Management, Risk Books, volume 2, chapter Credit Risk Modeling and Valuation: An Introduction. . 487–525.
7. Black F, Scholes MS (1973) The pricing of options and corporate liabilities. Journal of Political Economy 81: 637–54.
8. Merton RC (1974) On the Pricing of Corporate Debt: The Risk Structure of Interest Rates. Journal of Finance 29: 449–470.
9. Jarrow RA, Turnbull SM (1995) Pricing derivatives on financial securities subject to default risk. Journal of Finance 50: 53–86.
10. Jarrow RA, Lando D, Turnbull SM (1997) A markov model for the term structure of credit risk spreads. Review of Financial Studies 10: 481–523.
11. Duffie D, Singleton K (1999) Modeling the term structure of defaultable bonds. Review of Financial Studies 12: 687–720.
12. Hull JC, White A (2000) Valuing credit default swaps I: No counterparty default risk. Journal of Derivatives 8: 29–40.
13. Schönbucher PJ (2003) Credit Derivatives Pricing Models. New Jersey: John Wiley & Sons.
14. Black F, Cox JC (1976) Valuing Corporate Securities: Some Effects of Bond Indenture Provisions. Journal of Finance 31: 351–367.
15. Asvanunt A, Staal A (2009) The Corporate Default Probability model in Barclays Capital POINT platform (POINT CDP). Portfolio Modeling, Barclays Capital.
16. Asvanunt A, Staal A (2009) The POINT Conditional Recovery Rate (CRR) Model. Portfolio Modeling, Barclays Capital.
17. Katz YA, Shokhirev NV (2010) Default risk modeling beyond the first-passage approximation: Extended Black-Cox model. Physical Review E 82: 016116.
18. Rosenow B, Weissbach R (2009) Modelling correlations in credit portfolio risk. Journal of Risk Management in Financial Institutions 3: 16–30.
19. Gupton GM, Finger CC, Bhatia M (1997) CreditMetrics – Technical Document. Technical report, Morgan Guaranty Trust Company.
20. McKinsey & Company (1998) Credit Portfolio View, Approach Document und User's Manual. Technical report, McKinsey & Company.
21. Credit Suisse First Boston (1997) Credit Risk+: A Credit Risk Management Framework. Technical report, Credit Suisse First Boston (CSFB).
22. Schäfer R, Koivusalo A (2013) Dependence of defaults and recoveries in structural credit risk models. Economic Modelling 30: 1–9.
23. Koivusalo A, Schäfer R (2012) Calibration of structural and reduced–form recovery models. Journal of Credit Risk 8: 31–51.
24. Schönbucher PJ (2001) Factor models: Portfolio credit risks when defaults are correlated. Journal of Risk Finance 3: 45–56.
25. Glasserman P (2003) Tail approximations for portfolio credit risk. Working Paper.

26. Schäfer R, Sjölin M, Sundin A, Wolanski M, Guhr T (2007) Credit risk–a structural model with jumps and correlations. Physica A 383: 533–569.
27. Guhr T, Müller-Groeling A, Weidenmüller HA (1998) Random-matrix theories in quantum physics: common concepts. Physics Reports 299: 189–425.
28. Mehta M (2004) Random Matrices. 3rd edition. Academic Press
29. Becker A, Koivusalo A, Schäfer R (2012) Empirical evidence for the structural recovery model. arXiv:1203.3188.
30. Wishart J (1928) The Generalised Product Moment Distribution in Samples from a Normal Multivariate Population. Biometrika 20A: 32–52.
31. Olver FWJ (1974) Asymptotics and Special Functions Academic Press37–38.
32. Watson GN (1944) A Treatise on the Theory of Bessel Functions Cambridge University Press183.
33. Plerou V, Gopikrishnan P, Rosenow B, Amaral L, Stanley H (1999) Universal and nonuniversal properties of cross correlations in financial time series. Phys Rev Lett 83: 1471–1474.
34. Laloux L, Cizeau P, Bouchaud JP, Potters M (1999) Noise dressing of financial correlation matrices. Physical Review Letters 83: 1467–1470.
35. Potters M, Bouchaud J, Laloux L (2005) Financial applications of random matrix theory: Old laces and new pieces. Acta Physica Polonica B 36: 2767.
36. Burda Z, Jarosz A, Jurkiewicz J, Nowak M, Papp G, et al. (2011) Applying free random variables to random matrix analysis of financial data. Part I: A Gaussian case. Quantitative Finance 11: 1103.
37. Burda Z, Görlich AT, Waclaw B (2006) Spectral properties of empirical covariance matrices for data with power-law tails. Phys Rev E 74: 041129.
38. Biroli G, Bouchaud J, Potters M (2007) The Student ensemble of correlation matrices: Eigenvalue spectrum and Kullback-Leibler entropy. arXiv:0710.0802.
39. Abul-Magd AY, Akemann G, Vivo P (2009) Superstatistical generalizations of Wishart-Laguerre ensembles of random matrices. J Phys A 42: 175207.